Relationship between the magnitude of singular value and nonlinear stability*

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Abstract The relationship between the magnitude of singular value and nonlinear stability or instability of the basic flow is investigated. The results show that there is a good corresponding relationship between them. The magnitude of singular value decreases as the stability (or instability) of the basic flow increases (or decreases). In the stable case, the magnitude of the maximum singular value is much smaller than in the unstable case.

Keywords: **singular value, nonlinear stability, nonlinear instability.**

The motions in atmosphere or oceans are genuinely nonlinear. However, on some occasions, it is plausible and effective to simplify the nonlinear processes by linearizing them. Therefore, within a range of valid linearization approximation, singular values and singular vectors are widely utilized in theoretical research and practical implementations, such as the study of predictability[1] and ensemble forecast[2].

The magnitude of the maximum singular value represents the possibly fastest growing rate of the initial perturbations over a finite time interval, called the optimization time interval, with a given norm. Its corresponding singular vectors are the fastest growing initial perturbations. Clearly the concepts of singular value and singular vector are closely related to the linear stability and instability problems over a finite time interval[3]. On the other hand, there are quite a few definitions for nonlinear stability[4], one of which is as follows. If the growth rates of all the initial perturbations with a given norm cannot exceed a constant, no matter how long they evolve, the basic state is called nonlinearly stable[4~6]. This definition is widely utilized and the related stability problem is well-studied.

From the definitions of singular value and nonlinear stability, there should exist some relationship between them. In general, nonlinear processes are comparatively complicated. It is difficult to establish the criteria for nonlinear stability. However, it is relatively simple to find out singular values, at least by numerical approaches. If the relationship between the nonlinear stability and the singular values is clarified, some useful information about the nonlinear stability can be obtained from the singular values. In this paper, with the nonlinear stability criteria for some particular cases[5,6], the

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relationship between the singular value and the nonlinear stability is investigated. Using the two-dimensional barotropic quasigeostrophic model as an example, we examine the changes of the magnitudes of the maximum singular values versus the increasing(decreasing) nonlinear stability of the basic flows, with respect to energy norm. From the results of numerical experiments, there is significant difference in the magnitudes of the maximum singular values between the stable flows and unstable ones. Therefore, the magnitudes of singular values represent some characteristics of nonlinear stability and instability.

1 The model

The two-dimensional barotropic quasigeostrophic model is

$$\frac{\partial P}{\partial t} + \partial(\phi, P) = 0, \tag{1}$$

where ϕ is the streamfunction and $\partial(\phi, P)$ the two-dimension Jacobian. The potential vorticity is defined by

$$P = \nabla^2\phi - F\phi + f + \frac{f}{H}h_s, \tag{2}$$

where $F^{-1} = \frac{gH}{f^2}$ is the square of Rossby radius of deformation, f the Coriolis parameter, and h_s the topography. The flows are periodic in the west-east direction, and the north-south boundary is rigid. That is,

$$\left.\frac{\partial \phi}{\partial x}\right|_{y=0,2Y} = 0, \tag{3}$$

and the condition $\frac{\partial}{\partial t}\int_0^{2X} \left.\frac{\partial \phi}{\partial y}\right|_{y=0,2Y} dx = 0$ should also be satisfied.

Linearizing Eqs. (1) and (2), we can obtain its tangent linear model[7]. From the tangent linear model, its corresponding adjoint model can be obtained, then the singular values can be computed[8,9].

2 Numerical results

By the analysis of the numerical results, we will discuss the relationship between the magnitude of singular value and nonlinear stability with and without the topography being considered, with respect to energy norm.

2.1 Without the consideration of topography

We first study the case in which the topography is not considered. The basic flow is assumed to be[10]

$$\psi(y) = \psi_0\cos\left(\frac{\pi l y}{2Y} + \alpha\right), \tag{4}$$

where $\psi(y)$ is the streamfunction of the basic flow, with ψ_0 being the amplitudes of $\psi(y)$, l the wavenumber, Y the half of the north-south width, and α the displacement.

In this case, the criterion for nonlinear stability[5,6,10] is

$$l < 1. \tag{5}$$

Formula(5) shows that the stability of the basic flow is significantly effected by the wavenumber l. The stability decreases (or increases) as l becomes larger (or smaller). Furthermore, according to the study on validity of tangent linear model[10], the valid period for both stable and unstable basic flows under this condition can be as long as 2 days. Within a range of valid linearization approximation, some numerical experiments have been made to calculate the singular values. Figs. 1 and 2 show how the magnitudes of the maximum singular values change with the basic flows of different wavenumber l, over an optimal time interval of 24, 30, 36, 42, and 48 hours.

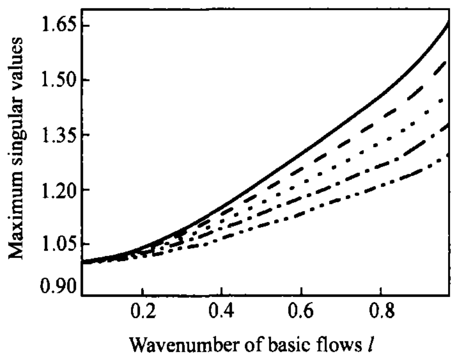

Fig. 1 The maximum singular values vs. the wavenumber of the nonlinearly stable basic flow, over a time interval of 48 h (solid), 42 h (dashed), 36 h (dot), 30 h (dash-dot), 24 h (dash-dot-dot).

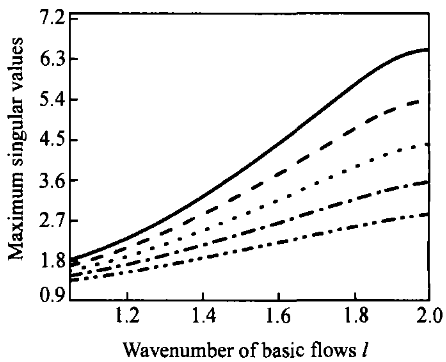

Fig. 2 The maximum singular values vs. the wavenumber of the nonlinearly unstable basic flow. The time intervals are the same as those in Fig. 1.

To study the difference in singular values between the stable and unstable basic flows, the averages of the maximum singular values for 20 stable cases and 20 unstable ones are indicated in Table 1.

Table 1 Averaged maximum singular values for stable and unstable cases*

Type	Time interval/h				
	24	30	36	42	48
Stable	1.118	1.155	1.190	1.228	1.265
Unstable	2.084	2.484	2.946	3.472	4.071

* Topography is not considered.

From Figs. 1 and 2 and Table 1, the following conclusions can be drawn:

(i) The magnitudes of the maximum singular values in the case of stable basic flow are much smaller than those in the case of unstable one.

(ii) The longer the time interval is, the larger the maximum singular value becomes, and the more distinguished difference in singular values between the stable and unstable cases is.

(iii) The magnitude of the maximum singular value is significantly influenced by the wavenumber l. It declines rapidly as l becomes smaller, that is, the stability increases.

2.2 With the consideration of topography

We then study the case in which the topography is expressed as

$$h(y) = h_0\left(\gamma \sin\left(\frac{\pi l y}{2Y}\right) + \gamma_2 \cos\left(\frac{\pi l y}{2Y}\right) + 1\right). \tag{6}$$

The basic flows[10] are chosen as

$$\psi(y) = \psi_0\left(\gamma_1 \sin\left(\frac{\pi l y}{2Y}\right) + \gamma_2 \cos\left(\frac{\pi l y}{2Y}\right) + A_1 \sin\left(\sqrt{\left(\frac{\pi l}{2Y}\right)^2 - \frac{f_0 h_0}{\psi_0}}\, y\right)\right.$$
$$\left. + A_2 \cos\left(\sqrt{\left(\frac{\pi l}{2Y}\right)^2 - \frac{f_0 h_0}{\psi_0}}\, y\right)\right), \tag{7}$$

where the meanings of $\psi(y)$, ψ_0, l and Y are the same as those in Sec. 2.1, $h(y)$ is the topography, h_0 the amplitude of the topography $h(y)$. γ_1, γ_2, A_1 and A_2 are the coefficients. The definition of other parameters can be found in Reference[10].

Under this condition, the criterion[5,6,10] for nonlinear stability is

$$0 < \left(\frac{\pi l}{2Y}\right)^2 + F - \frac{f_0 h_0}{\psi_0} < \left(\frac{\pi l}{2Y}\right)^2 + F. \tag{8}$$

From Refs. [5], [6] and [10], we know that the stability of the basic flow increases (or decreases) as the wavenumber l becomes smaller (or larger). Considering the validity of tangent linear model[10] which is 2 days under this condition, as in Sec. 2.1, we draw Figs. 3 and 4 to indicate the magnitudes of the maximum singular values varying with the basic flows of different wavenumber l, over a time interval of 24, 30, 36, 42, and 48 hours.

To manifest the difference between the stable basic flows and unstable ones, the averages for 20 stable and 20 unstable cases are presented in Table 2.

From Figs. 3 and 4 and Table 2, similar conclusions can be reached as those in Sec. 2.1. However, the difference between the stable and unstable cases is not so distinct as that in the above.

Table 2 Averaged maximum singular values for stable and unstable cases*

Type	Time interval/h				
	24	30	36	42	48
Stable	1.332	1.433	1.539	1.652	1.771
Unstable	1.795	2.062	2.355	2.680	3.028

* Topography is considered.

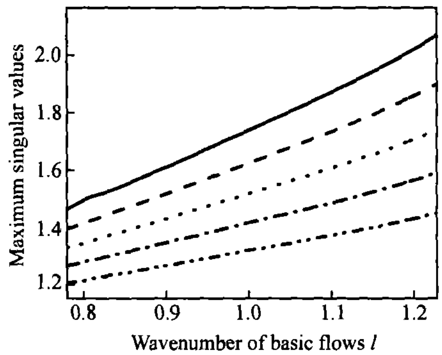

Fig. 3 The same as in Fig. 1, but the topography is considered.

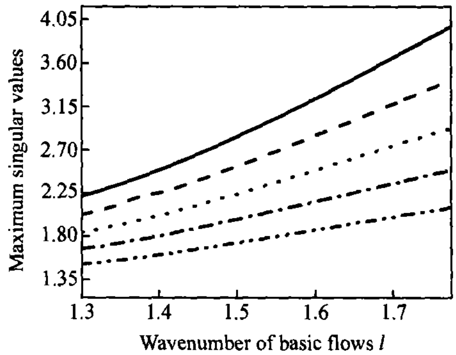

Fig. 4 The same as in Fig. 2, but the topography is considered.

3 Conclusions and discussions

The magnitudes of singular values are significantly affected by the nonlinear stability and instability of the basic flows. As the instability increases and time interval prolongs, the magnitude of the maximum singular value gets larger. The magnitude of the maximum singular value in the unstable case is much larger than that in the stable case. Furthermore, the difference becomes more distinct while the time interval is lengthened. This can be explained as follows. If the time interval is short, the perturbations cannot grow enough to distinguish the difference in the maximum singular values between unstable basic flows and stable ones.

References

1 Palmer, T. N. et al. Singular vectors and the predictability of weather and climate. Phil. Trans. R. Soc. Lond. A, 1994, 348: 459.

2 Buizza, R. Optimal perturbation time evolution and sensitivity of ensemble prediction to perturbation amplitude. Q. J. R. Meteorol. Soc., 1995, 121: 1705.

3 Molteni, F. et al. Predictability and finite-time instability of the northern winter circulation. Q. J. R. Meteorol. Soc., 1993, 119: 269.

4 Darry, D. H. et al. Nonlinear stability of fluid and plasma equilibria. Physics Reports, 1985, 123: 1.

5 Mu, M. Nonlinear stability of two-dimensional quasi-geostrophic motions. Geophys. Astrophys. Fluid Dyn., 1992, 65: 57.

6 Mu, M. et al. On Arnold's second nonlinear stability theorem for two-dimensional quasigeostrophic flow. Geophys. Astrophys. Fluid Dyn., 1994, 75: 21.

7 Lacarra, J. F. et al. Short-range evolution of small perturbations in a barotropic model. Tellus, 1988, 40A: 81.

8 Lorenz, E. N. A study of the predictability of a 28-variable atmospheric model. Tellus, 1965, XVII 3: 321.

9 Talagrand, O. et al. Variational assimilation of meteorological observations with the adjoint vorticity equation. Q. J. R. Meteorol. Soc., 1987, 113: 1311.

10 Mu, M. et al. The impact of nonlinear stability and instability on the validity of the tangent linear model. Adv. in Atmos. Sci., 2000, 17: 375.